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LETTER TO THE EDITOR

Quasi Potts model and bond dilution on non-alternate lattices

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Abstract. The q-state quasi Potts model is shown to be related in the $q = 1$ limit to the statistics of alternate clusters on non-alternate lattices.

Let us consider a q-state quasi Potts model (following the terminology of Wu (1982)) with Hamiltonian

$$-\beta\mathscr{H} = K \sum_{(ij)} \delta_q(n_i + n_j) \tag{1}$$

on a lattice $\mathscr{L}$ with N sites and coordination number z. $K = J/k_BT$ with $J > 0$ is the ferromagnetic coupling between nearest-neighbour sites, the sum is over the $N_b = Nz/2$ bonds (ij) of the lattice, n_i is a q-state Potts variable associated with site i ($n_i = 0, 1, \ldots, q-1$) and δ_q is a Kronecker delta function modulo q. This type of interaction was introduced by Enting (1975) in a generalised version of the Baxter–Wu model (see also Turban 1982).

On alternate lattices (1) has the same thermodynamics as the original Potts model (Potts 1952). In the partition function

$$Z_N = \sum_{\{n,n'\}} \exp\left(K \sum_{(ij)} \delta_q(n_i + n'_j)\right), \tag{2}$$

where the n'_j are associated with the sites of one of the two sublattices, let us make the change of variables

$$n_j = q - n'_j \qquad (\text{mod}\, q). \tag{3}$$

Then

$$Z_N = \sum_{\{n\}} \exp\left(K \sum_{(ij)} \delta_q(n_i - n_j)\right) \equiv \sum_{\{n\}} \exp\left(K \sum_{(ij)} \delta_{n_i,n_j}\right). \tag{4}$$

On non-alternate lattices the quasi Potts interaction of equation (1) leads to frustration effects. Consider for instance a triangular face with the variables (n_1, n_2, n_3); the three bonds are satisfied in the state $n_1 = n_2 = n_3 = 0$, otherwise at least one of the three bonds is frustrated $n_1 = n_2 = t$, $n_3 = q - t$ $(t \neq 0)$ except when $2t = q$, i.e. when q is even the state with $n_1 = n_2 = n_3 = q/2$ gives another unfrustrated configuration. It follows that in the Ising limit ($q = 2$) there is no frustration. To sum up, on non-alternate

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lattices the ground state is non-degenerate when q is odd and twofold degenerate when q is even.

The original Potts model in the $q = 1$ limit is known to be related to the percolation problem (Kasteleyn and Fortuin 1969, Wu 1978, Lubensky 1979); in the remainder of this letter we show that the quasi Potts model may be used to describe the statistics of alternate clusters on non-alternate lattices.

Let us rewrite the quasi Potts Hamiltonian on a non-alternate lattice $\mathscr{L}$ as

$$-\beta\mathscr{H} = K \sum_{(ij)} [q\delta_q(n_i + n_j) - 1] + H \sum_i [q\delta_q(n_i) - 1] \tag{5}$$

where the external field H favours the state $n_i = 0$ ($H > 0$). The partition function reads

$$Z_N = \sum_{\{n\}} \exp(-\beta\mathscr{H}) = \exp[(q-1)(KN_b + HN)]Z'_N \tag{6}$$

where

$$Z'_N = \sum_{\{n\}} \prod_{(ij)} \exp\{qK[\delta_q(n_i + n_j) - 1]\} \prod_i \exp\{qH[\delta_q(n_i) - 1]\}. \tag{7}$$

Using the identity

$$\exp\{qK[\delta_q(n_i + n_j) - 1]\} \equiv (1-p)\left(1 + \delta_q(n_i + n_j)\frac{p}{1-p}\right) \tag{8}$$

where $p = 1 - \exp(-qK)$ is a bond occupation probability, we get

$$Z'_N = (1-p)^{N_b} \sum_{\{n\}} \prod_{(ij)} \left(1 + \delta_q(n_i + n_j)\frac{p}{1-p}\right) \prod_i \exp\{qH[\delta_q(n_i) - 1]\}. \tag{9}$$

Expanding the product on the bonds, to each term in the expansion corresponds a subgraph $\mathscr{G}$ on $\mathscr{L}$ with a factor $\delta_q(n_i + n_j)p/(1-p)$ associated with any occupied bond (ij). Taking the trace over the Potts variables $\{n\}$, an alternate connected part of $\mathscr{G}$ with n_s sites and n_b bonds contributes a factor

$$[p/(1-p)]^{n_b}[1 + (q-1)\exp(-qHn_s)]. \tag{10}$$

Isolated site contributions are given by equation (10) with $n_s = 1$ and $n_b = 0$. For non-alternate clusters, when q is odd, the only non-vanishing contribution, due to frustration effects, is given by the state $n_i = 0$ (all i on the cluster) and reads

$$[p/(1-p)]^{n_b}. \tag{11}$$

When q is even there is a further contribution from the state $n_i = q/2$, giving

$$[p/(1-p)]^{n_b}[1 + \exp(-qHn_s)]. \tag{12}$$

Let $N\mathscr{K}_{a(na)}(\mathscr{G}, n_s, n_b)$ be the number of alternate (non-alternate) clusters with n_s sites and n_b bonds on $\mathscr{G}$,

$$N_b(\mathscr{G}) = \sum_{n_s, n_b} Nn_b[\mathscr{K}_a(\mathscr{G}, n_s, n_b) + \mathscr{K}_{na}(\mathscr{G}, n_s, n_b)] \tag{13}$$

the number of occupied bonds on $\mathscr{G}$ and

$$N\mathscr{K}_{a(na)}(\mathscr{G}, n_s) = \sum_{n_b} N\mathscr{K}_{a(na)}(\mathscr{G}, n_s, n_b) \tag{14}$$

the number of alternate (non-alternate) clusters with n_s sites on $\mathscr{G}$. Then

$$Z_N = \exp[N(q-1)(Kz/2+H)] \times \sum_{\mathscr{G}} P(\mathscr{G}) \prod_{n_s} [1+(q-1)\exp(-qHn_s)]^{N\mathscr{K}_a(\mathscr{G},n_s)} \qquad (q \text{ odd}) \tag{15}$$

where $P(\mathscr{G}) = p^{N_b(\mathscr{G})}(1-p)^{N_b - N_b(\mathscr{G})}$ is the probability of occurrence of $\mathscr{G}$. When q is even, a factor

$$\prod_{n_s} [1+\exp(-qHn_s)]^{N\mathscr{K}_{na}(\mathscr{G},n_s)} \tag{16}$$

has to be inserted in equation (15).

The odd version of the model given by equation (15) may be continued to non-integral values of q. Since $\ln(Z_N) = 0$ when $q = 1$, the free energy per site may be defined as

$$f(H) = \lim_{N\to\infty} \frac{\ln Z_N}{N(q-1)} = \lim_{N\to\infty} \frac{1}{N} \frac{\partial \ln Z_N}{\partial q}\bigg|_{q=1} \tag{17}$$

when $q = 1$. Or, using equation (15),

$$f(H) = Kz/2 + H + \lim_{N\to\infty} \sum_{\mathscr{G}} P(\mathscr{G}) \sum_{n_s} \mathscr{K}_a(\mathscr{G}, n_s)\, e^{-Hn_s}. \tag{18}$$

Let

$$\mathscr{K}_a(n_s) = \sum_{\mathscr{G}} P(\mathscr{G}) \mathscr{K}_a(\mathscr{G}, n_s) \tag{19}$$

be the mean number of alternate clusters per site with n_s sites; then

$$\lim_{H\to 0+} f(H) = Kz/2 + \sum_{n_s}{}' \mathscr{K}_a(n_s) \tag{20}$$

where the prime on the sum indicates that only finite clusters contribute in the thermodynamic limit. The free energy per site in zero external field is thus related to the mean number of alternate clusters per site. The order parameter

$$\frac{\partial f}{\partial H}\bigg|_{H\to 0+} = 1 - \sum_{n_s}{}' n_s \mathscr{K}_a(n_s) \tag{21}$$

does not give the percolation probability as usual (Wu 1978, Lubensky 1979). The sum rule

$$\mathscr{P}_a(p) + \sum_{n_s}{}' n_s \mathscr{K}_a(n_s) + \sum_{n_s} n_s \mathscr{K}_{na}(n_s) = 1, \tag{22}$$

where $\mathscr{P}_a(p)$ is the percolation probability for alternate clusters, states that any site belongs to a cluster, finite or infinite, alternate or non-alternate. It follows that

$$\frac{\partial f}{\partial H}\bigg|_{H\to 0+} = \mathscr{P}_a(p) + \sum_{n_s} n_s \mathscr{K}_{na}(n_s) \tag{23}$$

contains non-alternate cluster contributions. The susceptibility

$$\chi = \frac{\partial^2 f}{\partial H^2}\bigg|_{H\to 0+} = \sum_{n_s}{}' n_s^2 \mathscr{K}_a(n_s) \tag{24}$$

gives the mean square cluster size for alternate clusters.

The statistics of alternate clusters on non-alternate lattices is relevant to the problem of bond dilution in antiferromagnetically coupled spin systems, since non-alternate clusters are frustrated whereas alternate clusters may sustain antiferromagnetic order. The quasi Potts model on non-alternate lattices is by itself of interest with its dependence on the parity of q and the unusual effect of frustration which here reduces the ground-state degeneracy.

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